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The Curvature of a Single Contraction Operator on a Hilbert Space *

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Abstract

This note studies Arveson's curvature invariant for d-contractions $T = (T_1, T_2, \ldots, T_d)$ for the special case $d = 1$, referring to a single contraction operator T on a Hilbert space. It establishes a formula which gives an easy-to-understand meaning for the curvature of a single contraction. The formula is applied to give an example of an operator with nonintegral curvature. Under the additional hypothesis that the single contraction T be "pure", we show that its curvature $K(T)$ is given by $K(T) = -\text{index}\,(T) := -(\dim\ker(T) - \dim\text{coker}\,(T))$.

1 The curvature of a single operator

This note studies Arveson's curvature invariant for d-contractions $T = (T_1, T_2, \ldots, T_d)$ for the special case $d = 1$, referring to a single contraction operator T on a Hilbert space. It establishes a formula which gives an easy-to-understand meaning for the curvature of a single contraction. The formula is applied to give an example of an operator with nonintegral curvature. Under the additional hypothesis that the single contraction T be "pure", we show that its curvature $K(T)$ (defined below) is given by $K(T) = -\text{index}\,(T) := -(\dim\ker(T) - \dim\text{coker}\,(T))$.

Let T be a contraction operator on a Hilbert space H, and $\Delta_T := \sqrt{1 - TT^*}$. Assume that Δ_T has finite rank. Then the curvature $K(T)$ of T (our shorthand

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for what should properly be called the curvature of the Hilbert module associated with T) is defined in [2] as:

$$K(T) := \int_{|z|=1} dz \lim_{r\uparrow 1} (1-r^2)\,\mathrm{tr}\,(\Delta_T (1-rzT^*)^{-1}(1-r\bar{z}T)^{-1}\Delta_T) \quad . \tag{1}$$

This is a specialization to the case of a single operator of Arveson's more general theory of d-contractions, which are finite sets of d commuting operators satisfying an auxiliary condition analogous to contractiveness of our T.

We refer the reader to [1], [2], and [3] for the definition and basic properties of d-contractions. However, we consider d-contractions for $d > 1$ solely for purposes of placing our results within the framework of the more general theory, and essentially no knowledge of d-contractions is necessary to follow our proofs. The only reliance on the general theory is that Arveson's Stability of Curvature result, [3], Section 3, Corollary 1, is used in the proof of Proposition 1. However, as noted there, the reader can easily establish this result directly for the special case of a 1-contraction, which is all that we need.

The definition of curvature implicitly assumes the existence of the limit in (1). A theorem stated in [2], and proved in [3] (Theorem A), guarantees the existence of the limit for almost all z, and moreover bounds it above by the rank of Δ_T. For the case of a single operator, this also follows from the discussion of [7], Chapter VI, Section 1, particularly, page 238, equation (1.5).

Let $T : H \to H$ be a contraction on a Hilbert space H with rank $\sqrt{1-TT^*}$ finite. Note that this implies that range $\sqrt{1-TT^*}$ = range $(1-TT^*)$, a fact which will be used frequently without comment.

First we associate with T a partial isometry Q with the same curvature, so that for most purposes of computing the curvature, we may assume that T is itself a partial isometry. This is not always necessary, but it makes many problems easier to think about.

Proposition 1 *With T as just described, set*

$$Q := \begin{bmatrix} T & \sqrt{1-TT^*} \\ 0 & 0 \end{bmatrix} \quad ,$$

considered as an operator on $H \oplus \mathrm{range}\,(1-TT^*)$.

Then Q is a partial isometry with $K(Q) = K(T)$.

Moreover, $\mathrm{rank}\,(1-QQ^*) = \mathrm{rank}\,(1-TT^*)$*,*
and, $\mathrm{rank}\,(1-Q^*Q) = \mathrm{rank}\,(1-T^*T)$*.*

Proof: That $K(Q) = K(T)$ follows from one of Arveson's key results for d-contractions, Stability of Curvature, [3], Section 3, Corollary 1. For our case of a 1-contraction, a proof can alternatively be obtained by a straightforward calculation of $K(Q)$, based on its definition (1).

Since

$$1 - QQ^* = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{Range}\,(1-TT^*)} \end{bmatrix} \quad , \tag{2}$$

it is obvious that Q is a partial isometry with rank $(1-QQ^*) = $ rank $(1-TT^*)$.

Next we show that rank $(1-Q^*Q) = $ rank $(1-T^*T)$. We have

$$1 - Q^*Q = \begin{bmatrix} 1-T^*T & -T^*\sqrt{1-TT^*} \\ -\sqrt{1-TT^*}T & TT^* \end{bmatrix} \quad .$$

Since the rank of an operator matrix is at least as large as the rank of any entry, if rank $(1-T^*T)$ is infinite, so is rank $(1-Q^*Q)$. Thus we may assume that rank $(1-T^*T)$ is finite.

Let $C_i,\ \ i=1,2$, denote the i'th column of the matrix for $1-Q^*Q$, considered in the obvious way as operators, e.g., $C_1 : H \to$ range $(1-TT^*)$. Then

$$C_2\sqrt{1-TT^*} = \begin{bmatrix} -T^*(1-TT^*) \\ TT^*\sqrt{1-TT^*} \end{bmatrix} = \begin{bmatrix} -(1-T^*T)T^* \\ \sqrt{1-TT^*}TT^* \end{bmatrix} = -C_1T^*$$

Since the domain of C_2 is range $\sqrt{1-TT^*}$, this implies that range $C_2 \subset$ range C_1, and hence range $(1-Q^*Q) = $ range C_1.

It is well known (e.g., [8], Section 147) that $\sqrt{1-TT^*}\,T = T\sqrt{1-T^*T}$, so

$$C_1 = \begin{bmatrix} (1-T^*T) \\ -T\sqrt{1-T^*T} \end{bmatrix} = \begin{bmatrix} \sqrt{1-T^*T} \\ -T \end{bmatrix} \sqrt{1-T^*T}$$

Since

$$\begin{bmatrix} \sqrt{1-T^*T} \\ -T \end{bmatrix}$$

is an isometry, the map $\sqrt{1-T^*T}x \mapsto C_1x,\ \ x \in H$, defines an isometric bijection between range $\sqrt{1-T^*T}$ and range $C_1 = $ range $(1-Q^*Q)$. Hence rank $(1-T^*T) = $ rank $(1-Q^*Q)$. ∎

Next we derive a simple formula for $K(Q)$, along with a variant formula for $K(T)$ which does not mention Q. The formula for $K(Q)$ seems particularly helpful in thinking about these problems.

Theorem 2 *Let Q be a partial isometry such that $\Delta_Q := \sqrt{1-QQ^*}$ has finite rank, and let $e_1, e_2, \ldots, e_q$ be an orthonormal basis for* range (Δ_Q). *Then*

$$K(Q) = \sum_{k=1}^{q} \lim_{n\to\infty} \|Q^n e_k\|^2 \quad .$$

Moreover, for any contraction T for which Δ_T has finite rank,

$$K(T) = \lim_{n\to\infty} \operatorname{tr}\left({T^*}^n T^n (1-TT^*)\right) \quad .$$

Proof:

The boundedness of the integrand of the curvature justifies application of the Lebesgue Dominated Convergence Theorem to interchange limit and integral in

the definition (1) of curvature:

$$
\begin{aligned}
K(Q) \quad &:= \quad \int_{|z|=1} \lim_{r\uparrow 1} (1-r^2) \sum_{k=1}^{q} \langle (1-rzQ^*)^{-1} e_k, (1-r\bar{z}Q)^{-1} \rangle e_k \rangle \, dz && (3) \\
&= \quad \sum_{k=1}^{q} \lim_{r\uparrow 1} (1-r^2) \int_{|z|=1} \langle (1-r\bar{z}Q)^{-1} e_k, (1-r\bar{z}Q)^{-1} e_k \rangle \, dz && (4) \\
&= \quad \sum_{k=1}^{q} \lim_{r\uparrow 1} (1-r^2) \int_{|z|=1} \langle \sum_{i=0}^{\infty} (r\bar{z}Q)^i e_k, \ \sum_{j=0}^{\infty} (r\bar{z}Q)^j e_k \rangle \, dz && (5) \\
&= \quad \sum_{k=1}^{q} \lim_{r\uparrow 1} (1-r^2) \int_{|z|=1} \sum_{i,j=0}^{\infty} r^{i+j} z^{j-i} \langle Q^i e_k, \ Q^j e_k \rangle \, dz && (6) \\
&= \quad \sum_{k=1}^{q} \lim_{r\uparrow 1} (1-r^2) \sum_{i=0}^{\infty} r^{2i} \, \|Q^i e_k\|^2 && (7) \\
&= \quad \sum_{k=1}^{q} \lim_{i\to\infty} \|Q^i e_k\|^2 && (8)
\end{aligned}
$$

Equation (7) was obtained by interchanging the infinite sum and integration. This is justified because for fixed r, the infinite sum converges absolutely with sum of absolute values bounded above by $(1-r^2)^{-2}$.

Equation (8) is justified as follows. For fixed k, consider the decreasing sequence

$$1 \geq \|Qe_k\| \geq \|Q^2 e_k\| \geq \ldots \geq \lim_{i\to\infty} \|Q^i e_k\| \quad ,$$

and set $L := \lim_{i\to\infty} \|Q^i e_k\|^2$. Then for any positive integer m,

$$
\begin{aligned}
L \quad &= \quad L \lim_{r\uparrow 1} (1-r^2) \sum_{i=0}^{\infty} r^{2i} \\
&\leq \quad \lim_{r\uparrow 1} (1-r^2) \sum_{i=0}^{\infty} r^{2i} \, \|Q^i e_k\|^2 \\
&= \quad \lim_{r\uparrow 1} (1-r^2) \sum_{i=m}^{\infty} r^{2i} \|Q^i e_k\|^2 \\
&\leq \quad \|Q^m e_k\|^2 \, \lim_{r\uparrow 1} (1-r^2) \sum_{i=m}^{\infty} r^{2i} \\
&= \quad \|Q^m e_k\|^2 \quad .
\end{aligned}
$$

For sufficiently large m, the right side is arbitrarily close to L, showing that

$$\lim_{r\uparrow 1} (1-r^2) \sum_{i=0}^{\infty} r^{2i} \|Q^i e_k\|^2 = \lim_{i\to\infty} \|Q^i e_k\|^2 \quad ,$$

thus proving (8).

This proves the asserted formula for $K(Q)$. To prove the alternative formula for $K(T)$, define Q to be the partial isometry of Proposition 1 with $K(Q) = K(T)$. Recall from (2) that

$$\Delta_Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad ,$$

and check that

$$Q^n = \begin{bmatrix} T^n & T^{n-1}\Delta_T \\ 0 & 0 \end{bmatrix} \quad . \tag{9}$$

The formula for $K(T)$ follows immediately upon combining these observations, Proposition 1, the formula just proved for $K(Q)$, and the cyclic property of the trace:

$$\begin{aligned} K(T) &= K(Q) = \sum_{k=1}^{q} \lim_{n\to\infty} \|Q^n e_k\|^2 \\ &= \sum_{k=1}^{q} \lim_{n\to\infty} \langle \Delta_T T^{*n} T^n \Delta_T e_k, e_k \rangle \\ &= \lim_{n\to\infty} \operatorname{tr}(\Delta_T T^{*n} T^n \Delta_T) \\ &= \lim_{n\to\infty} \operatorname{tr}(T^{*n} T^n (1 - TT^*)) \end{aligned}$$

∎

A simple sufficient condition for the curvature to vanish is an immediate corollary:

Corollary 3 *Any contraction T whose positive powers T^n converge strongly to 0 has vanishing curvature: $K(T) = 0$.*

2 Relation to Arveson's curvature formula

Arveson [2] established a different formula for the curvature of a d-contraction T. Specialized to the case $d = 1$, it reads:

$$K(T) = \lim_{n\to\infty} \frac{\operatorname{tr}(1 - T^n T^{*n})}{n} \quad . \tag{10}$$

In order to make clear how our formula fits into Arveson's framework, we now derive ours assuming his. However, the resulting proof is not notably simpler than the direct proof above, and Arveson's proof is even more involved, corresponding to the fact that the case $d > 1$ is probably fundamentally more difficult than $d = 1$. For the single operator case $d = 1$, Arveson's formula follows similarly from ours.

Let T be a contraction with Δ_T of finite rank, and $e_1, \ldots, e_q$ an orthonormal basis for range $(\Delta_T) =$ range $(1 - TT^*)$. First note the collapsing sum:

$$1 - T^n T^{*n} = \sum_{i=0}^{n-1} T^i (1 - TT^*) T^{*i} \quad .$$

Hence

$$\begin{aligned} \lim_{n\to\infty} \frac{\operatorname{tr}(1 - T^n T^{*n})}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{tr}(T^{*i} T^i (1 - TT^*)) \\ &= \lim_{i\to\infty} \operatorname{tr}(T^{*i} T^i (1 - TT^*)) \quad . \end{aligned}$$

The last equality was obtained as follows. Consider the sequence

$$\begin{aligned} a_i &:= \operatorname{tr}(T^{*i} T^i (1 - TT^*)) \\ &= \operatorname{tr}((T^i \Delta_T)^* (T^i \Delta_T)) \\ &= \sum_{k=1}^{q} \langle\ (T^i \Delta_T)^* (T^i \Delta_T)) e_k, e_k \rangle \\ &= \sum_{k=1}^{q} \| T^i \Delta_T e_k \|^2 \quad . \end{aligned}$$

The last expression makes clear that $a_1 \geq a_2 \geq \ldots \geq 0$, so that the sequence has a limit $L = \lim_{i\to\infty} a_i$. We shall show that for any such sequence a_i,

$$\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} a_i = L \quad .$$

Since the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} a_i \right\}$ is bounded above by a_0, it is enough to show that its only possible accumulation point is L.

For any fixed m and all $n \geq m$,

$$L \leq \frac{1}{n} \sum_{i=0}^{n-1} a_i \leq \frac{1}{n} \sum_{i-0}^{m-1} a_i + \frac{n-m}{n} a_m \quad .$$

Letting n tend to infinity with m fixed, we see that any accumulation point of the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} a_i \right\}$ must lie between L and a_m. Finally, letting m tend to infinity shows that L is the only accumulation point. .

3 A simple formula for the curvature of a single, pure, contraction.

A single contraction T on a Hilbert space H will be called *pure* if for all $h \in H$, $\lim_{n\to\infty} T^{*n} h = 0$; i.e., if the adjoint powers T^{*n} converge strongly to 0. This

is the specialization to the case $d = 1$ of Arveson's more complicated definition of a pure d-contraction.

Arveson remarked [2] that it is generally difficult to determine the curvature of a d-contraction, but that "in the few cases where the computations can be explicitly carried out, the curvature turns out to be an integer." This led him to ask [2] if the curvature of a pure d-contraction need always be an integer.

This was a surprising suggestion, because nothing in the definition of curvature suggests that it should be an integer. Subsequently, D. Greene, S. Richter, and C. Sundberg [6] proved that indeed the curvature of any pure d-contraction is an integer. However, their function-theoretic methods do not seem to give an effective procedure for calculating this integer in particular cases, and a geometric, operator-theoretic interpretation of the curvature of a general d-contraction remains elusive as of this writing.

Our contribution toward understanding the meaning of the curvature invariant is a simple, usually easily computable, formula for the curvature of single, pure contraction; i.e., the special case $d = 1$. It states that the curvature is the difference of the dimensions of two subspaces, and hence is obviously integral. The methods of proof are operator-theoretic, based on unitary dilation theory as set forth in [7]. It uses neither the Greene/Richter/Sundberg result nor their function-theoretic methods, and thus gives an independent proof of their result for the special case $d = 1$.

Our characterization of the curvature of a single pure contraction is:

Theorem 4 *Let T be a* pure *contraction operator such that $\Delta_T := \sqrt{1 - TT^*}$ has finite rank. Then its curvature $K(T)$ is the integer*

$$K(T) = \dim \operatorname{range}(1 - TT^*) - \dim \operatorname{range}(1 - T^*T) \quad . \tag{11}$$

A counterexample in the next section uses Theorem 2 to show that the hypothesis that T be pure is essential.

Before proving the theorem, we review some standard facts about unitary dilations. Proofs can be found in [7], particularly Chapters 1, 2, and 6. We give specific references from this work for key facts required by the proof.

Let T be a contraction on a Hilbert space H, and U its minimal unitary dilation to a larger Hilbert space $K \supset H$. This means that $P_H U^n | H = T^n$ for all $n \geq 0$, where P_H denotes the projection to H, and minimality means that $K = \bigvee_{n=-\infty}^{\infty} U^n H$.

1. The minimal unitary dilation U for T may be constructed as follows. Define

$$K := \ldots \oplus \overline{\Delta_T H} \oplus \overline{\Delta_T H} \oplus H \oplus \overline{\Delta_{T^*} H} \oplus \overline{\Delta_{T^*} H} \ldots , \tag{12}$$

where the overscore denotes closure. (The closures turn out to be unnecessary in our context, but that only becomes apparent later.) Consider H as embedded in K in the obvious way. Then U is defined on K by:

$$\begin{aligned} U(\ldots,\ b_2,\ b_1,\ b_0, \boxed{h}, a_0,\ a_1,\ a_2,\ \ldots) &:= \\ (\ldots,\ b_2,\ b_1, \boxed{Th + \Delta_T b_0},\ -T^* b_0 + \Delta_{T^*} h,\ a_0,\ a_1,\ \ldots). \end{aligned} \tag{13}$$

Here $b_i \in \overline{\Delta_T H}$, $a_i \in \overline{\Delta_{T^*} H}$, and zero'th components (vectors in H) are distinguished by boxes.

The realization of U just given is best for some purposes, but a change of notation will bring out more clearly the features which will be important to us. Set $\mathcal{L} := (U - T)H$ and $\mathcal{L}_* := (U^* - T^*)H$. Informally, $\mathcal{L}$ is the leftmost $\overline{\Delta_{T^*} H}$ factor in (12). The other $\overline{\Delta_{T^*} H}$ factors are images of the leftmost under positive powers of U. Similarly, $\mathcal{L}_*$ is the rightmost $\overline{\Delta_T H}$ factor, and the other $\overline{\Delta_T H}$ factors are images of it under negative powers of U.

To reflect these insights, instead of realizing K as above, think of it as follows:

$$K \cong \ldots \oplus U^{-2}\mathcal{L}_* \oplus U^{-1}\mathcal{L}_* \oplus \mathcal{L}_* \oplus H \oplus \mathcal{L} \oplus U\mathcal{L} \oplus U^2\mathcal{L} \ldots \tag{14}$$

Here $\cong$ stands for unitary equivalence.

The conceptual advantage of (14) is that it makes clear at a glance much of the action of U on K. Unfortunately, it is awkward for the purpose of defining U due to logical circularity.

Embedded in U are two bilateral shifts which interact in a complicated way. One shifts $\mathcal{L}$, and the other shifts $\mathcal{L}_*$. One half of each shift is transparently visible in (14). For example, U obviously acts as a unilateral shift (with multiplicity $\dim \mathcal{L}$) on the invariant subspace

$$M(\mathcal{L})^+ := \bigoplus_{n=0}^{\infty} U^n \mathcal{L} \quad . \tag{15}$$

Since all iterates $U^n\mathcal{L}$, $-\infty \leq n \leq \infty$ are easily seen to be pairwise orthogonal, also U acts as a bilateral shift on the invariant subspace

$$M(\mathcal{L}) := \bigoplus_{n=-\infty}^{\infty} U^n \mathcal{L} \quad , \tag{16}$$

but the left half of this subspace, $M(\mathcal{L}) \ominus M(\mathcal{L})^+$, is embedded in a non-transparent way in K.

A subspace $\mathcal{S}$ such that the subspaces $U^n\mathcal{S}$ are pairwise orthogonal, $-\infty < n < \infty$, is called a *wandering subspace* for U. Thus $\mathcal{L}$ is a wandering subspace, and so is $\mathcal{L}_*$. For any wandering subspace $\mathcal{S}$, we'll use the notation $M(\mathcal{S})$ as defined in (16) with $\mathcal{L}$ replaced by $\mathcal{S}$.

2. The contraction T is pure, i.e., $T^{*n} \to 0$ strongly, if and only if

$$K = M(\mathcal{L}_*) := \bigoplus_{n=-\infty}^{\infty} U^n \mathcal{L}_*$$

(Chap. 2, Thm. 1.1, p. 57).

3. If T is pure, then $\dim \mathcal{L} \leq \dim \mathcal{L}_*$; equivalently, $\operatorname{rank} \Delta_{T^*} \leq \operatorname{rank} \Delta_T$. In particular, under our hypotheses that T is pure with Δ_T of finite rank, also Δ_{T^*} has finite rank, and both $\mathcal{L}$ and $\mathcal{L}_*$ are finite dimensional.

   This follows from item 2 above combined with [7], Chap. 1, Prop. 2.1, p. 4. Alternatively, it can be obtained for the case that we'll need, $\dim \mathcal{L}_* < \infty$, from the Reciprocity Lemma 5 below with $\mathcal{L}' := \mathcal{L}_*$. Assuming temporarily that $\dim \mathcal{L}$ is known to be finite, the Reciprocity Lemma applies as follows:

$$\dim \mathcal{L} = \operatorname{tr}(P_{\mathcal{L}}) = \operatorname{tr}(P_{\mathcal{L}} P_{M(\mathcal{L}_*)}) = \operatorname{tr}(P_{\mathcal{L}_*} P_{M(\mathcal{L})}) \leq \operatorname{tr}(P_{\mathcal{L}_*}) = \dim \mathcal{L}_* \ .$$

   The case of an infinite-dimensional $\mathcal{L}$ can be ruled out by applying the same reasoning with $\mathcal{L}$ replaced by finite-dimensional subspaces of $\mathcal{L}$.

4. When T is a partial isometry,

$$U\mathcal{L}_* = \overline{\Delta_T H} \quad .$$

   In particular, $U\mathcal{L}_* \subset H$.

   This is immediate from (13) after recalling that a partial isometry T satisfies $T^*(1 - TT^*)H = \{0\}$.

Let E and F be projections on a Hilbert space, at least one of which has finite rank. Then $\operatorname{tr}(EF) = \operatorname{tr}(E^2F) = \operatorname{tr}(EFE)$, so $\operatorname{tr}(EF)$ is always nonnegative, is zero if and only if E and F have orthogonal ranges, and takes on its maximum value $\dim(E)$ or $\dim\ (F)$ only when $E \leq F$ or $F \leq E$. Thus $\operatorname{tr}(EF)$ serves as a measure of how nearly the ranges of E and F coincide. For lack of a standard term, call $\operatorname{tr}(EF)$ the *affinity* between the ranges of E and F.

The following lemma, which we call the Reciprocity Lemma, may have some interest in its own right. It states that for wandering subspaces $\mathcal{L}$ and $\mathcal{L}'$ for a unitary operator U, the affinity between $\mathcal{L}$ and the closed span of the iterates $U^n\mathcal{L}'$, $-\infty \leq n \leq \infty$ is invariant under interchange of $\mathcal{L}$ and $\mathcal{L}'$.

Lemma 5 [Reciprocity Lemma] *Let $\mathcal{L}$ and $\mathcal{L}'$ be finite dimensional wandering subspaces for a unitary operator U on a Hilbert space K, and set*

$$M(\mathcal{L}) := \bigoplus_{n=-\infty}^{\infty} U^n\mathcal{L} \qquad \text{and} \qquad M(\mathcal{L}') := \bigoplus_{n=-\infty}^{\infty} U^n\mathcal{L}' \quad .$$

Then, denoting by P_S the projection on an arbitrary subspace S of K,

$$\operatorname{tr}(P_{\mathcal{L}} P_{M(\mathcal{L}')}) = \operatorname{tr}(P_{\mathcal{L}'} P_{M(\mathcal{L})}) \quad .$$

Proof:

Note that

$$P_{M(\mathcal{L})} = \sum_{n=-\infty}^{\infty} P_{U^n\mathcal{L}} = \sum_{n=-\infty}^{\infty} U^n P_{\mathcal{L}} U^{-n} \quad , \tag{17}$$

the sums converging in the strong operator topology.

Multiply (17) by $P_{\mathcal{L}'}$ on the left, take the trace of both sides, and suppose we can justify an interchange of sum and trace, obtaining

$$\begin{aligned} \mathrm{tr}\,(P_{\mathcal{L}'} P_{M_{\mathcal{L}}}) &= \mathrm{tr}\,(\sum_{n=-\infty}^{\infty} P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n}) \\ &= \sum_{n=-\infty}^{\infty} \mathrm{tr}\,(P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n}) \quad . \end{aligned} \tag{18}$$

Then the following simple calculation establishes the lemma:

$$\begin{aligned} \mathrm{tr}\,(P_{\mathcal{L}'} P_{M(\mathcal{L})}) &= \sum_{n=-\infty}^{\infty} \mathrm{tr}\,(P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n}) \\ &= \sum_{n} \mathrm{tr}\,(P_{\mathcal{L}} U^{-n} P_{\mathcal{L}'} U^n) \\ &= \mathrm{tr}\,(P_{\mathcal{L}} P_{M(\mathcal{L}')}) \quad , \end{aligned} \tag{19}$$

where the last line was obtained from (18) with $\mathcal{L}$ and $\mathcal{L}'$ interchanged and the summation index n replaced by $-n$.

The interchange of sum and trace required to justify the above calculation is not immediate because the trace is not continuous in the strong operator topology. However, the trace is well-known to be a *normal* linear functional, which implies that for any increasing sequence of trace class positive operators

$$A_1 \leq A_2 \leq \ldots \leq A_m \leq \ldots$$

converging in the strong operator topology to a trace class operator A, we have

$$\lim_{m\to\infty} \mathrm{tr}\,(A_m) = \mathrm{tr}\,(A) \quad . \tag{20}$$

This property is a slight specialization of the definition of normality. It follows routinely from the definition $\mathrm{tr}\, A := \sum_{i=1}^{\infty} \langle Ae_i, e_i \rangle$, with $\{e_i\}$ an orthonormal basis.

Noting that

$$\mathrm{tr}\,(P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n}) = \mathrm{tr}\,(P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n} P_{\mathcal{L}'})$$

and applying (20) with $A_m := \sum_{n=-m}^{m} \mathrm{tr}\,(P_{\mathcal{L}'} U^n P_{\mathcal{L}} U^{-n} P_{\mathcal{L}'})$ proves (18).

I thank W. Arveson for suggesting the above proof to replace the unattractive direct calculation of an earlier draft. ∎

Proof of Theorem 4:

Proposition 1 shows that we may assume that T is a partial isometry. We are going to use Theorem 2 to calculate $K(T)$ by calculating

$$\lim_{n\to\infty} \|T^n e\|^2$$

for $e \in \Delta_T H$. For any $h \in H$,

$$\|T^n h\|^2 = \|P_H U^n h\|^2 \quad .$$

Since $U^n h \in H \bigoplus M(\mathcal{L})^+ = H \bigoplus (\bigoplus_{k=0}^{\infty} U^k \mathcal{L})$,

$$\begin{aligned} \|T^n h\|^2 &= \|U^n h\|^2 - \|P_{M(\mathcal{L})^+} U^n h\|^2 \\ &= \|h\|^2 - \|P_{M(\mathcal{L})^+} U^n h\|^2 \quad . \end{aligned} \tag{21}$$

Write $K = M(\mathcal{L}) \bigoplus R$, where (as always), the direct sum denotes an orthogonal direct sum, so this defines the subspace R, which reduces K because $M(\mathcal{L})$ does. Then any $k \in K$ can be written

$$k = \sum_{i=-\infty}^{\infty} U^i f_i + r \quad ,$$

with $f_i \in \mathcal{L}$ and $r \in R$. And, for any $h \in H$,

$$h = \sum_{i=-\infty}^{-1} U^i f_i + r \quad . \tag{22}$$

Substituting (22) in (21) gives:

$$\begin{aligned} \lim_{n\to\infty} \|T^n h\|^2 &= \|h\|^2 - \lim_{n\to\infty} \sum_{i=-n}^{-1} \|f_i\|^2 \\ &= \|h\|^2 - \|P_{M(\mathcal{L})} h\|^2 \\ &= \|h\|^2 - \langle P_{M(\mathcal{L})} h,\ h\rangle \quad . \end{aligned} \tag{23}$$

Choose an orthonormal basis $e_1, \ldots, e_q$ for range $\Delta_T H$. Then substituting (23) in Theorem 2 gives:

$$\begin{aligned} K(T) &= \sum_{i=1}^{q} \lim_{n\to\infty} \|T^n e_i\|^2 \\ &= q - \sum_{i=1}^{q} \langle P_{M(\mathcal{L})} e_i,\ e_i\rangle \\ &= q - \operatorname{tr}(P_{M(\mathcal{L})} P_{\Delta_T H}) \quad . \end{aligned} \tag{24}$$

Item 2 remarked that for a partial isometry T, $\Delta_T H = U\mathcal{L}_*$, and substituting this in (24) gives:

$$K(T) = q - \operatorname{tr}(P_{M(\mathcal{L})} P_{U\mathcal{L}_*}) \quad . \tag{25}$$

By the Reciprocity Lemma 5,

$$\operatorname{tr}(P_{M(\mathcal{L})} P_{U\mathcal{L}_*}) = \operatorname{tr}(P_{M(U\mathcal{L}_*)} P_{\mathcal{L}}) \quad .$$

But obviously,

$$M(U\mathcal{L}_*) := \bigoplus_{n=-\infty}^{\infty} U^n U\mathcal{L}_* = \bigoplus_{k=-\infty}^{\infty} U^k \mathcal{L}_* = M(\mathcal{L}_*) \quad .$$

Combining these facts gives the desired conclusion:

$$\begin{aligned} K(T) &= q - \operatorname{tr}(P_{M(\mathcal{L})} P_{U\mathcal{L}_*}) = q - \operatorname{tr}(P_{M(U\mathcal{L}_*)} P_{\mathcal{L}}) \\ &= q - \operatorname{tr}(P_{M(\mathcal{L}_*)} P_{\mathcal{L}}) = q - \operatorname{tr}(P_{\mathcal{L}}) \\ &= \dim \operatorname{range} \Delta_T - \dim \operatorname{range} \Delta_{T^*} \\ &= \dim \operatorname{range} (1 - TT^*) - \dim \operatorname{range} (1 - T^*T) \quad . \end{aligned}$$

The second line follows from item 2's observation that the hypothesis that T be pure is equivalent to $M(\mathcal{L}_*) = K$. ∎

Recall that a Fredholm operator T is one with closed range and finite-dimensional kernel and cokernel (denoted $\ker(T)$ and $\operatorname{coker}(T) := \ker(T^*)$). The *index* of a Fredholm operator T is defined by

$$\operatorname{index}(T) := \dim \ker(T) - \dim \operatorname{coker} T \quad . \tag{26}$$

A fundamental theorem (e.g., [5], p. 128, Thm. 5.36) states that the index is invariant under compact perturbations: for any Fredholm operator T and compact operator C, $T + C$ is Fredholm, and $\operatorname{index}(T + C) = \operatorname{index}(T)$.

Formula (26) makes sense when T has finite-dimensional kernel and cokernel even if T doesn't have closed range. However, since the closed range hypothesis is needed to prove the fundamental theorem just mentioned, the term "index" is generally restricted to Fredholm operators. Nevertheless, for purposes of the present exposition, it will be convenient to broaden the definition of $\operatorname{index}(T)$ to include cases in which T has finite-dimensional kernel and cokernel, but not necessarily closed range.

When told of the curvature formula (11) given by Theorem 4, W. Arveson remarked that it looked something like an operator index and that he had been working on a conjecture that under appropriate hypotheses, the curvature of a d-contraction would be the index of an associated operator which he calls D_+, reminiscent of the Dirac operator. Shortly thereafter, he wrote up these results in [4], which proves this for d-contractions whose associated Hilbert modules are finite rank, pure, and graded, in the terminology of [3]. It asks if the "graded" hypothesis can be removed, and also if the associated operator D_+ necessarily has closed range (and so is Fredholm).

For the case of a 1-contraction, the associated operator D_+ is unitarily equivalent to T. Corollary 6 below observes that under the hypotheses of Theorem 4, T is Fredholm, and its curvature equals $-\operatorname{index}(T)$. The differences between Corollary 6 and the specialization of Arveson's result to the single operator case are that the closed range property is proved for $d = 1$, and the "graded"

hypothesis is not needed. This holds out hope that the "graded" and "closed range" hypotheses might be removable for d-contractions with $d > 1$.

The interest in identifying the curvature with an index, apart from its evident aesthetic appeal, is that the index is stable under compact perturbations, but the curvature is not known to possess such stability. The strongest result along these lines known as of this writing is [3], Corollary 1, Stability of Curvature, which proves stability of the curvature under certain special finite rank perturbations. Arveson [4] notes that removing the "closed range" hypothesis would establish a much stronger stability of curvature result, and removing the "graded" hypothesis would strengthen it further.

Corollary 6 *Let T be an operator satisfying the hypotheses of Theorem 4. Then T is Fredholm, and*

$$K(T) = -\mathrm{index}\,(T) \quad .$$

Proof: Let T be an operator on a Hilbert space H satisfying the hypotheses of Theorem 4. First we sketch the simple proof that T must be Fredholm.

The assumed finiteness of the rank of $1-TT^*$ implies that $\mathrm{coker}\,(T)$ is finite-dimensional. Since we have already noted that $\mathrm{rank}\,(1-T^*T) \leq \mathrm{rank}\,(1-TT^*)$, also $\ker(T)$ is finite-dimensional. That T must have closed range under these circumstances can be easily seen by noting that closed range is equivalent to a gap above 0 in the spectrum of T^*T. If there were not such a gap, then $1-T^*T$ would not have finite rank.

To show that (11) equals $-\mathrm{index}\,(T)$, let $\tilde{T}$ be the operator on $H \oplus \mathrm{range}\,(1-TT^*)$ defined by the operator matrix:

$$\tilde{T} := \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \quad .$$

Let Q be the partial isometry

$$Q := \begin{bmatrix} T & \sqrt{1-TT^*} \\ 0 & 0 \end{bmatrix}$$

of Proposition 1. Since Q is a compact perturbation of $\tilde{T}$,

$$\mathrm{index}\,(Q) = \mathrm{index}\,(\tilde{T}) = \mathrm{index}\,(T) \quad .$$

Also, since Q is a partial isometry, $\dim\ker(Q) = \dim\mathrm{range}\,(1-Q^*Q)$ and $\dim\mathrm{coker}\,(Q) = \dim\mathrm{range}\,(1-QQ^*)$. Hence by Proposition 1,

$$\begin{aligned} \mathrm{index}\,(T) = \mathrm{index}\,(Q) &:= \dim\ker(Q) - \dim\mathrm{coker}\,(Q) \\ &= \dim\mathrm{range}\,(1-Q^*Q) - \dim\mathrm{range}\,(1-QQ^*) \\ &= \dim\mathrm{range}\,(1-T^*T) - \dim\mathrm{range}\,(1-TT^*) \\ &= -K(T) \quad . \end{aligned}$$

∎

Remark: The above proof that index $(T) = \dim \operatorname{range}(1-T^*T) - \dim \operatorname{range}(1-TT^*)$ is concise and natural within our context, but may not be the most insightful. A slightly messier but more straightforward proof can be based on the well-known fact that for any operator T, the restriction of T^*T to its initial space (defined as the orthogonal complement of its nullspace) is unitarily equivalent to the restriction of TT^* to the closure of *its* initial space. (The equivalence can be implemented by the partial isometry U in the polar decomposition $T = U\sqrt{T^*T}$ restricted to its initial space.) From this it follows that any nonzero eigenvalue for T^*T is also an eigenvalue for TT^*, with the same multiplicity, so that in the expression $\dim \operatorname{range}(1 - T^*T) - \dim \operatorname{range}(1 - TT^*)$, the dimensions of the eigenspaces corresponding to nonzero eigenvalues cancel, leaving the only contribution to this expression as $\dim \ker(T) - \dim \ker(T^*) = \operatorname{index}(T)$.

4 A contraction with non-integral curvature

Now we apply Theorem 2 to construct a simple example of an operator with non-integral curvature, in fact with arbitrary real curvature $\kappa \geq 0$. This shows that Theorem 4's hypothesis that T be pure cannot be omitted. I have been told that the existence of non-pure contractions with non-integral curvatures was implicitly known or expected by experts in the field, so the interest of the example may lie more in its simplicity than novelty.

It is enough to produce a partial isometry Q with range Δ_Q spanned by a single unit vector e satisfying

$$\lim_{n\to\infty} \|Q^n e\|^2 = \kappa \quad .$$

First suppose $0 \leq \kappa \leq 1$, and set $\lambda := \sqrt{1-\kappa}$.

Let T be the bilateral weighted shift defined on an orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ by:

$$Te_n := \begin{cases} e_{n+1} & \text{if } n \neq 0 \\ \lambda e_1 & \text{if } n = 0 \end{cases} \quad .$$

Then one routinely computes that $\Delta_T := \sqrt{1-TT^*}$ is the rank 1 operator whose only non-zero eigenvalue is $\sqrt{1-\lambda^2} = \sqrt{\kappa}$, with corresponding eigenvector e_1.

Let Q be the associated partial isometry given by Proposition 1. We may realize Q as acting on a space with orthonormal basis $\{e_\infty\} \bigcup \{e_n\}_{n=-\infty}^{\infty}$ obtained by adjoining a new unit vector named e_∞ to the previous orthonormal basis for the space on which T was defined. Then Q is defined by $Qe_n := Te_n$ for n finite, and $Qe_\infty := \sqrt{1-\lambda^2}\, e_1 = \sqrt{\kappa}\, e_1$.

As in Proposition 1, one routinely computes that $\Delta_Q := \sqrt{1-QQ^*}$ is the one-dimensional projection with range spanned by e_∞. From Theorem 2

$$K(Q) = \lim_{n\to\infty} \|Q^n e_\infty\|^2 = \lim_{n\to\infty} \|T^{n-1}\sqrt{\kappa}\, e_1\|^2 = \kappa \quad .$$

This shows that any κ with $0 \leq \kappa \leq 1$ can be the curvature of some contraction. To see that any real number can be the curvature of some contraction,

first check that curvature is additive over direct sums: for any two contractions T_1, T_2, we have

$$K(T_1 \oplus T_2) = K(T_1) + K(T_2) \quad .$$

This follows routinely from the original definition (1) of curvature, or slightly more easily, from Theorem 2. Then any desired non-negative real curvature can be obtained by direct summing appropriate copies of the above example.

References

[1] Arveson, W., Subalgebras of C^*-algebras III: Multivariable operator theory, Acta Math. **181** (1998), 159-228

[2] Arveson, W., The curvature of a Hilbert module over $C[z_1, \dots, z_d]$, Proc. Natl. Acad. Sci. USA, **96** (1999), 11096-11099

[3] Arveson, W., The curvature invariant of a Hilbert module over $C[z_1, \dots, z_d]$, preprint, revised 17 February, 1999, to appear in J. für Reine und Angew. Mat. **522** (2000)

[4] Arveson, W., The Dirac operator of a commuting d-tuple, preprint, May 22, 2000

[5] Douglas, R., *Banach Algebra Techniques in Operator Theory*, Academic Press, 1972

[6] Greene, D., Richter, S., and Sundberg, C., The structure of inner multipliers on spaces with complete Nevalinna Pick kernels, preprint

[7] Sz.-Nagy, B., and Foias, C., *Harmonic Analysis of Operators on Hilbert Space* (revised English edition), North-Holland, Amsterdam, 1970

[8] Riesz, F., and Sz.-Nagy, B., *Functional Analysis*, N.Y., 1955